

Lectures

Robust Design Optimization and Catastrophe Theory

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Abstract

There exist many examples for optimization in CAE covering a broad range and also combining different physical disciplines. Goals are for example the reduction of weight and/or the increase of efficiency. Often these optimizations tend towards optimized designs with a lower safety margin. Therefore proving the robustness of the design is becoming increasingly important. Typically methods that are applied for checking the robustness of designs are robustness evaluations or reliability analysis. These analyses then also include the scattering behavior of input parameters. The combination of these techniques leads to the framework of CAE based Robust Design Optimization.

Catastrophe theory originated as a special branch of dynamical systems theory by the French mathematician Rene Thom in the 1960s. Catastrophes are bifurcations between different equilibria. Catastrophe theory studies and classifies phenomena characterized by sudden shift in the behavior caused by small changes for the basic parameters.

These sudden changes are also well known in real world system applications, however the underlying fundamental parameters are often not that evident nor is it clear how to translate it into simulations for virtual prototyping.

There are different classifications of the catastrophes based on the parameters of the potential functions involved, for example the seven elementary types of catastrophes described by Thom (Thom 1989) or the ADE classification by Arnold (Arnold 1992).

As parametric robust design optimization also studies the variation of input parameters and their influence on the equilibrium of the target function, the idea of this paper is to study the elementary functions of catastrophe theory like folding, cusp or swallowtail by robust design optimization techniques, to gain deeper insights into system behavior combining robust design optimization and catastrophe theory.

In catastrophe theory the input variables are distinguished between control variables and state variables. In our examination we found it very useful to handle the control variables as the input parameters for the RDO analysis and to use the state variables to construct signals (like for example time-series, operating lines, characteristic curves used in this paper as functions of one variable) and to examine for examples the number of minima of these signals.

Typically the number of minima changes at the bifurcation set. With that approach we are able to classify the different domains in control space. Also the spatial difference of the extrema along a signal is a useful indicator that can be selected as a robustness criteria that is changing smoothly to zero, before the sudden change happens.

1. Robust Design Optimization

CAE-based optimization is well established in engineering. With the increase of parametric models and new technologies there is an increase of feasible designs and there are many parametric optimization techniques available. For an optimization at least one goal needs to be defined. This goal can be for example the reduction of weight or the increase in performance. In the simplest case the variation of a single parameter can lead to an optimization. In parametric optimization we are searching for minima of at least one function f , dependent on a set of m input parameters:

$$f = f(x_1, x_2, \dots, x_m)$$

For the minimum it is sufficient that the first derivative $f' = 0$ and the second derivative $f'' > 0$. In catastrophe theory the focus is on areas where also $f'' = 0$, as this is the area of the bifurcations.

As target functions often still are flexible during the early phases of the development process and with regards to try to understand the design space it is more relevant to study the relations between the input variables x_1, x_2, \dots, x_m and the output variables y_1, y_2, \dots, y_l .

Real-world applications usually depend on many – often conflicting - parameters, filtering the most important ones is a critical task. This is usually done first in a sensitivity analysis, before starting with the optimization. Global sensitivity analysis (Saltelli et al. 2008) that are using DOE (Design of Experiments) approaches to scan the design space have shown to be very efficient, especially if using the latin hypercube sampling for many input parameters (Most et al. 2011).

The sensitivity analysis is not only enabling to scan the design space but it also enables to quantify the influence of input variables with respect to output variables, fields. The coefficient of prognosis (CoP) has proven to be a very good quantification of these influences in many real world applications (Most et al. 2011). The CoP reaches far beyond best fitting towards high predictability, enabling the Meta model of Optimal Prognosis (MOP).

For our application to catastrophe theory it is important to note that the output can be also not only variables but also fields. This is the case for example, when the output is a signal. Furthermore the output can be derived quantities of the signal, like the number of peaks in a certain range.

A product should of course not only be optimal under one possible set of parameter realizations. It also has to function with sufficient reliability under scattering environmental conditions. In the virtual world we can proof that e.g. with a stochastic analysis, which leads to CAE-based robustness evaluation. If CAE-based optimization and robustness evaluation is combined, we are entering the area of Robust Design Optimization (RDO) which is also called Design for Six Sigma (DFSS) or just Robust Design (RD).

The main idea behind that methodology is that uncertainties are considered in the design process. These uncertainties may have different sources like tolerances of the geometrical dimensions and material properties caused by production or by deterioration. Some of these uncertainties may have a significant impact to the design characteristics which has to be considered in the design optimization procedure.

While for the optimization the variation of one parameter may lead to an “improved” result, the task in RDO is much more challenging: we have to take care as reasonable as possible that no important variable has been neglected.

Two main approaches for RDO are the variance and the reliability based robustness evaluation. Today, the majority of RDO approaches uses variance-based robustness measures in order to minimize the variation of a response with or without constraining the mean, which is known as Taguchi approach, or to reach a certain level of safety quantified by the safety margin or sigma levels, where Design for Six Sigma is one possible concept.

In the framework of variance based RDO not only to target for designs with low standard deviation is of interest, but it is of importance also to consider a sufficient safety margin to failure or operation limits (Most et al. 2015). As a consequence the proof of a sufficient safety margin to failure and operational limits should be the primary goal of RDO in virtual prototyping. In case of high requirements with respect to the safety level like in Design for Six Sigma a final reliability proof should be considered.

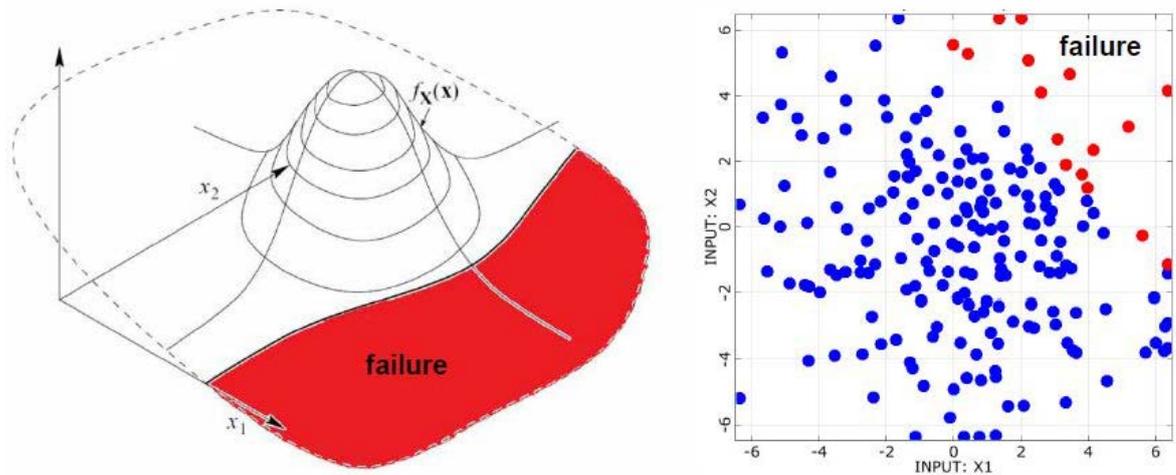


Figure 1: Reliability analysis as multi-dimensional integration of the probability density of the inputs over the failure domain (left) and integration by Monte Carlo Simulation (right)

Within the reliability method the probability of reaching a failure limit is obtained by an integration of the probability density of the uncertainties in the failure domain. One well-known method is the Monte Carlo Simulation, which can be applied independently of the model non-linearity and the number of input parameters. This method is very robust and can detect several failure regions with highly non-linear dependencies. Unfortunately, it requires an extremely large number of solver runs to proof rare events.

Therefore, more advanced sampling strategies have been developed like Directional Sampling, where the domain of input variables is scanned by a line search in different directions, or Importance Sampling, where the sampling density is adapted in order to cover the failure domain sufficiently and to obtain more accurate probability estimates with much less solver calls. Other methods like the First or Second Order Reliability Method are still more efficient than the sampling methods by approximating the boundary between the safe and the failure domain, the so-called limit state. In contrast to a global low order approximation of the whole response, the approximation of the limit state around the most probable failure point is much more accurate. Nevertheless, only one dominant failure point can be found and evaluated. A good overview of these “classical” methods is given in (Bucher 2009).

2. Catastrophe Theory

Catastrophes are bifurcations between different equilibria (including the possibility of no equilibria). Catastrophe theory studies and classifies phenomena characterized by sudden shift in the behavior caused by small changes for the basic parameters. The input parameters are separated into state variables and control parameters. It is the small change of the control parameters that can have a large effect on the stability of the systems. A simple case is the elementary catastrophe of the fold:

$$y = x^3 + ux$$

x is the state variable and u is the control variable. Differentiating $y(x)$:

$$y' = 3x^2 + u$$

$$y'' = 6x$$

We see that the function $y(x)$ has for negative u a minimum at

$$x = \sqrt{-u/3}$$

and no extremum at all for positive u .

Assuming for negative u we have a generic stable system with this equilibrium at the minima, this system will change in an abrupt way if u crosses to positive values.

The theorem of Thom (Thom 1989) about the seven (not counting Morse function x^2 as this simple one has no co-dimension) elementary catastrophes states that:

Any smooth function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, having a critical point at the origin, and having co-dimension ≤ 4 , is right equivalent (up to sign) to one of the following:

Table 1: The elementary catastrophes

(Morse)	x^2
Fold	$x^3 + ux$
Cusp	$x^4 + ux^2 + vx$
Swallowtail	$x^5 + ux^3 + vx^2 + wx$
Butterfly	$x^6 + tx^4 + ux^3 + vx^2 + wx$
Elliptic Umbilic	$x^3 - xy^2 + w(x^2 + y^2) + ux + vy$
Hyperbolic Umbilic	$x^3 + y^3 + wxy + ux + vy$
Parabolic Umbilic	$y^4 + x^2y + wx^2 + ty^2 + ux + vy$

Here t, u, v, w are control variables, x and y are state variables.

For any elementary catastrophe the function described with these control variables –the unfolding - is the most general description around the origin. There are no additional control parameters needed for the description of this elementary catastrophe. As we seek to assure such a completeness of the important input variables in RDO it is worthwhile to consider it here a little bit more in detail. This completeness of the unfolding is also known as structural stability.

Consider as an example the fold bifurcation $x^2 + u = 0$ already discussed above, but now studying the effects of some small perturbation ε

$$x^2 + u + \varepsilon = 0$$

This only shifts the bifurcation point from $u = 0$ to $u = -\varepsilon$, but the structure of the bifurcation remains the same.

Now consider the pitchfork bifurcation $x^3 + ux = 0$. This equation can be derived from the cusp, but missing one control parameter v , so this is not an unfolded bifurcation set. If we now add some small perturbation:

$$x^3 + ux + \varepsilon = 0,$$

we see that the pitchfork separates into two separate branches (Figure 2), therefore the pitchfork bifurcation is not structurally stable!

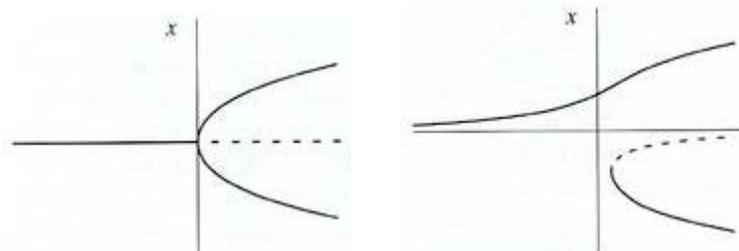


Figure 2: Pitchfork bifurcation with $\varepsilon = 0$ (left image) and effect of $\varepsilon > 0$ (right image)

So the generic approach to understand the bifurcation set of the possible design space is to reach for a minimal set of control parameters so that we have structural stability of the bifurcation set. If we have for example too few control parameters we may not be aware of whole classes of different solutions.

In this paper we will focus the study of the fold, cusp, swallow tail and butterfly with the means of RDO.

The intention is to work out generic RDO approaches that can be applied in more complicated situations to detect system changes. This will be addressed in the next chapter.

Next we discuss the cusp:

$$y = x^4 + ux^2 + v$$

The cusp has the equilibrium set M

$$4x^3 + 2ux + v = 0$$

The set of singularities is the subset of M with:

$$12x^2 + 2u = 0$$

Eliminating x from the two equations we get:

$$8u^3 + 27v^2 = 0$$

This bifurcation set, the cusp (cmp Figure 3), is known for a lot of real world examples, like caustics in a coffee cup or for civil engineering the behavior of the maximum load for small deviations from symmetry if applying a load force to the euler arch (Poston et al. 1998).

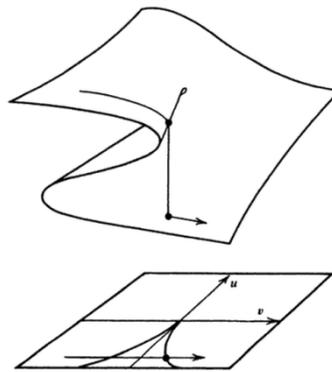


Figure 3: The cusp

The potential of the swallow tail is:

$$V(x) = x^5 + ux^3 + vx^2 + wx$$

With three control parameters u,v and w.

The equilibrium hyper surface M is described by:

$$5x^4 + 3ux^2 + 2vx + w = 0$$

And the singularity subset B of M:

$$20x^3 + 6ux + 2v = 0$$

From the last two equations it is possible to eliminate x, however for the study of the quantitative behavior of B it is favorable to consider slices of B with

constant values u within the control space C (spanned by u, v and w). If we consider a surface C_u with constant u and B_u the intersection of B with C_u a curve discussion of B_u (Saunders 1980) leads to two different cases $u > 0$ and $u < 0$ (Figure 4). The figure for $u < 0$ leads to the name swallow tail.

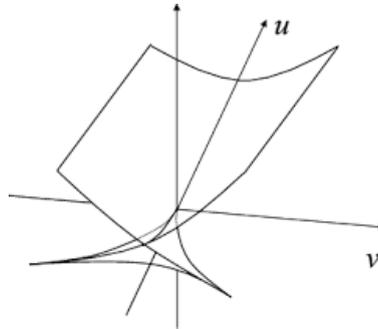


Figure 4: The bifurcation set of the swallow tail catastrophe function in control space (u, v, w)

Similar for the elementary catastrophe named butterfly:

$$V(x) = x^6 + tx^4 + ux^3 + vx^2 + wx$$

The equilibrium hyper surface M is described by:

$$6x^5 + 4tx^3 + 3ux^2 + 2vx + w = 0$$

And the singularity subset B of M :

$$30x^4 + 12tx^2 + 6ux + 2v = 0$$

The usual approach is to consider subsets of B with constant t and u . Especially for $u = 0$ and for negative t the characteristic bifurcation subset shown in (Figure 5) shows up.

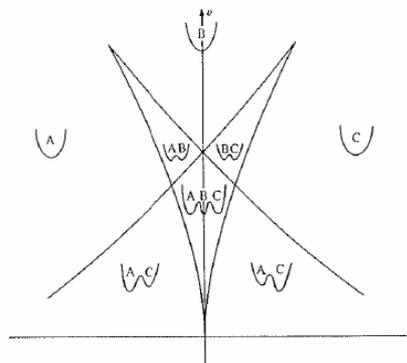


Figure 5: The bifurcation set of the butterfly catastrophe function in sub space (w, v) for $u=0, t < 0$

3. Some applications of RDO to Catastrophe Theory

In the following we used the software optiSLang and the methods described in (Dynardo 2016) to study the four elementary catastrophes fold, cusp, swallow tail and butterfly with RDO techniques.

The fold:

The results of a simple sensitivity analysis are shown in Figure 6:

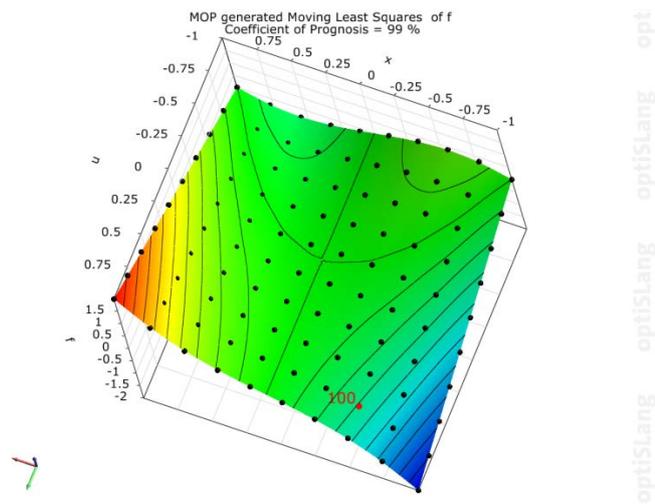


Figure 6: The MOP generated from a sensitivity analysis of the fold shows that for negative constant u , we have a minima and a maxima.

We see directly the bifurcation leading for negative u to a minima and a maxima. Of course this extrema can be followed up in the optimization, like for example in Figure 7 we show the results of a Simplex optimization keeping constant $u = -2$:

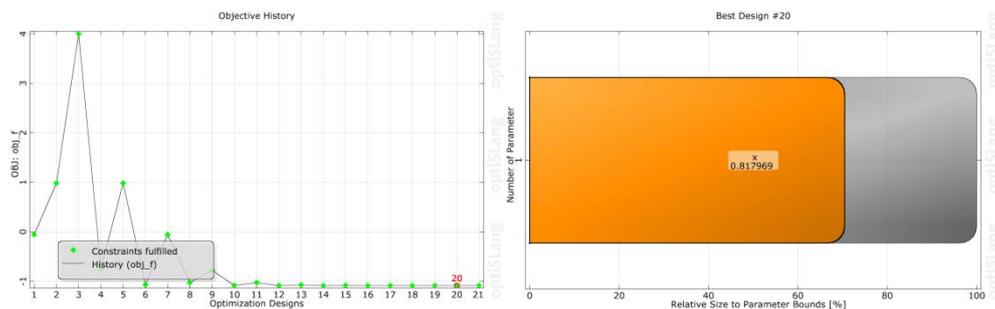


Figure 7: Optimization run using the Simplex optimization algorithm for constant $u = -2$

We are more interested to develop methods that help to detect critical points like the one at $x = 0$, $u = 0$ already when approaching this point. For this we take the following approach: for constant u we consider $f(x)$ as a signal. This

signal has for negative u a minima and a maxima. These two extrema coincide at $u = 0$. So we take the distance between the minima and the maxima of the signal as a measure for the robustness. Figure 8 shows how the signal and some derived responses are defined.

ID	Description	Type	Value	Expression	Responses
1	x_vector	REAL VECTOR	[201]	linspace(-2,2,201)	diff_x_peaks 0
2	y_vector	REAL VECTOR	[201]	x_vector^3+u*x_vector	diff_y_peaks 0
3	xy_signal	REAL XYDA...	[1:201]	xydata(x_vector,y_vector)	min_gradient 0.0004
4	min_gradient	REAL	0.0004	min(derive(xy_signal, FORWARD))[1]	num_peaks 0
5	num_peaks	REAL	0	#if min_gradient > 0 #then 0 #else size(extract(peaks(xy_signal),0))	x_vector [201]
6	diff_x_peaks	REAL	0	#if num_peaks > 1 #then abs((extract(peaks(xy_signal),0))[0]-(extract(peaks(xy_signal),0))[1]) #else 0	xy_signal [1:201]
7	diff_y_peaks	REAL	0	#if num_peaks > 1 #then abs((extract(peaks(xy_signal),1))[0]-(extract(peaks(xy_signal),1))[1]) #else 0	y_vector [201]

Figure 8: Definition of signals and derived values

Figure 9 shows the result for the output variable diff_x_peaks (the difference in x between the minima and the maxima) that coming from negative u goes to zero at $u = 0$.

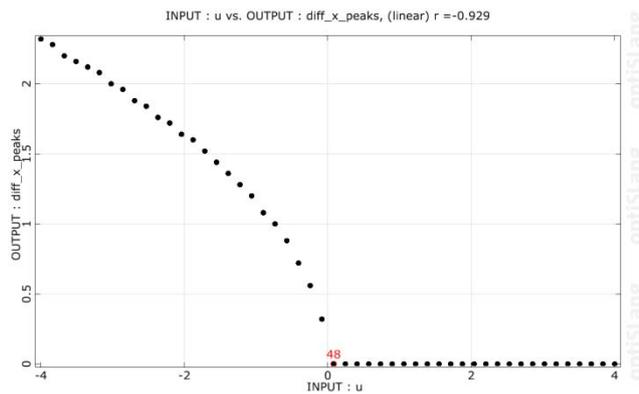


Figure 9: Difference between the extrema as a function of u .

This approach can be generalized to quite arbitrary signals having many extrema and therefore used as a generic approach in studying signals.

The cusp:

For this elementary catastrophe function the bifurcation set, the cusp, is where the functions change from one to two minima (or the other way round). This can be directly seen from the results of the sensitivity analysis shown in Figure 10 for $v = 0$, that is the path directly through the top of the cusp.

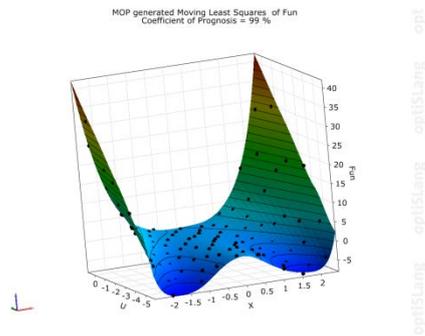


Figure 10: Metamodel of the cusp with $v = 0$

We see that for the negative constant u there are two minima.

Setting the bifurcation set itself as the limit state, we can use direct robustness evaluations to show the different areas in control space:

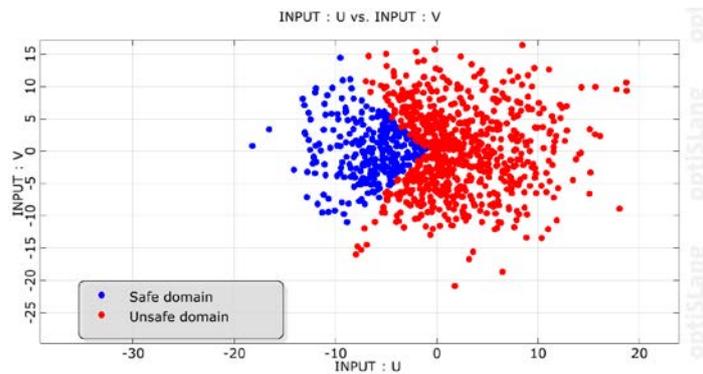


Figure 11: Robustness Analysis of the cusp

We take the region with two minima as the safe domain. For this purpose we used the cusp bifurcation set to describe the limit state in an analytic way as a function of u and v . In a more generic approach we can take the number of minimal peaks for the limit state to detect the border lines. For this approach we used the directional sampling reliability method. The result is shown in Figure 12:

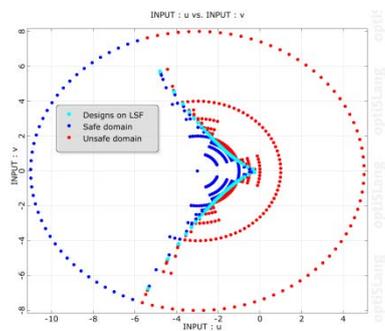


Figure 12: Directional sampling used to detect the cusp

The swallow tail:

We continue this path of distinguishing the different regions by their number of peaks. Figure 13 show an anthill plot in 3D control space, using the colour for the different number of peaks.

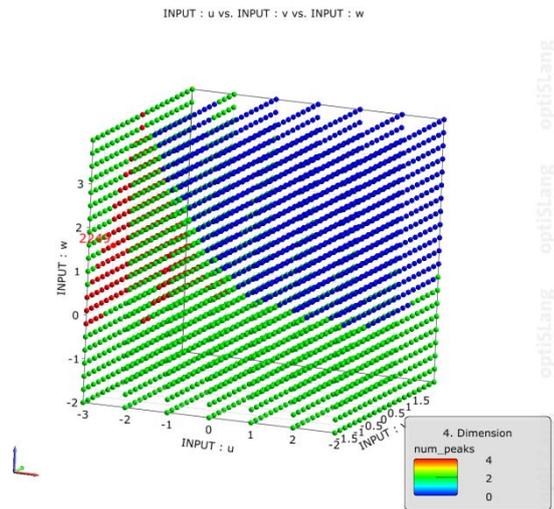


Figure 13: Three dimensional anthill plot, different colour is used for a different number of peaks

If we take a two-dimensional sub space, with a constant negative u, the swallow tail becomes more evident:

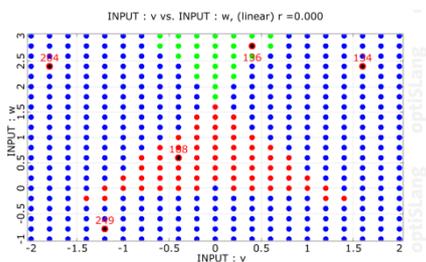


Figure 14: Anthill plot of the swallow tail regions, selecting five designs

The selected five designs are shown in figure 15 (green: no minima, blue: one, red: two)

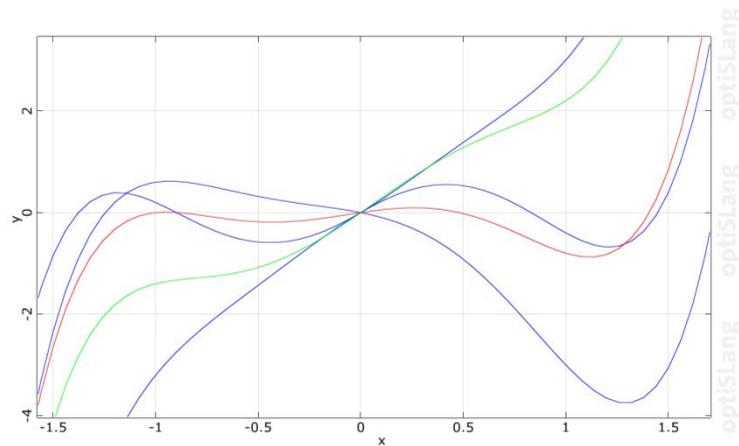


Figure 15: The five designs selected in Figure 14

The butterfly:

The control parameter space is four dimensional. Checking the design points directly for the minimum number of peaks, we can present the results in a 5D cloud plot, using here u , v , t as spatial coordinates, the number of minima is coloured and w is associated to the size representation of the design point. This leads to Figure 16:

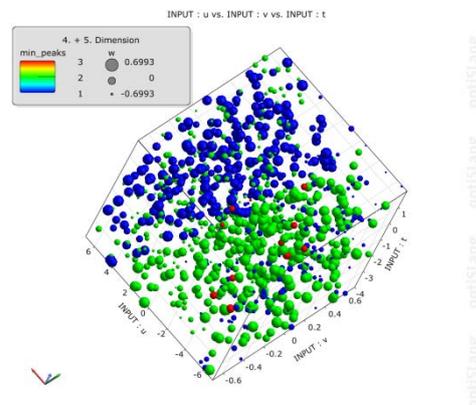


Figure 16: Cloud plot of design points, the red points belong to signals with three minima

This presentation techniques can give some hints, but it is also useful to look for appropriate sub space as for example in the classical discussion with $u = 0$, $t < 0$ we can use for example again the directional sampling to detect the different bifurcation sets, as for example in Figure 17 we have defined as the safe region a region where there is only one minima:

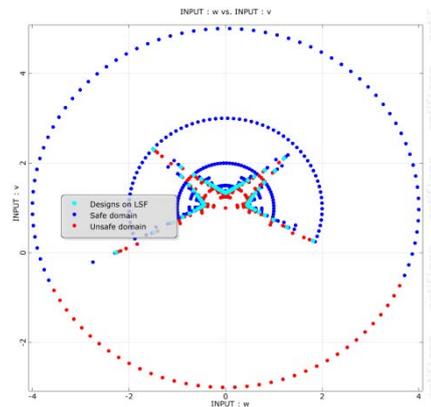


Figure 17: Directional sampling for $t < 0$, $u = 0$

So the number of minima from the signals show again to be a valuable output parameter for distinguishing the different domains where the border is the bifurcation set.

4. Physical Applications

We apply the RDO techniques to two standard examples, one from optics and one from structural mechanics. These examples are discussed in many of the textbooks of catastrophe theory (Saunders 1980; chapter 5). At least to our knowledge this is the first time that RDO techniques are applied to these examples from catastrophe theory. Still the curves that are treated as signals are derived from analytical functions and we like to extend this approach in the near future to signals derived from CAE simulations and/or measurements.

Optics

Caustics are surfaces, points where the rays are concentrated and focused. This concentration leads to high intensities (the name caustic itself is derived from the greek word for burned: $\kappa\alpha\upsilon\sigma\tau\acute{o}\varsigma$)

For example the reflection of the sunlight in automobile headlights by their reflectors can cause severe heat problems inside the headlights. Sensitivity analysis with thermo-mechanical CAE simulations are used to study these influences. It may be quite useful, for example to reduce the number of necessary simulation runs, if the caustics are known. This can be achieved with RDO techniques for example defining the caustic as the limit state function.

Berry showed that caustics are bifurcation sets (Berry 1976). As caustics are in general surfaces in \mathbb{R}^3 there are up to 3 control variables. Therefore structural stable caustics will show similarity to fold, cusp, swallow tail and to elliptic or hyperbolic umbilic.

A caustic that can be easily seen in a nearly full cup of coffee caused by sunlight is shown in Figure 18 a). Figure 18 b) shows the cusp like form of the caustic by using ray tracing.

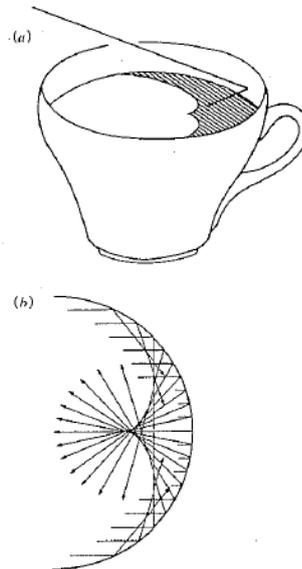


Figure 18: Example of a cusp like caustic in a nearly full coffee cup (Saunders, 1980)

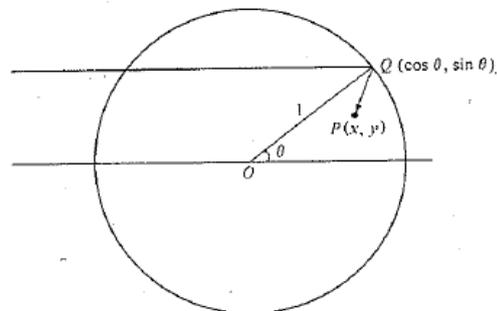


Figure 19: A ray from a distance d reflected by Q to P

If we calculate the length of the way from a distance d reflected by Q to any point $P(x,y)$, according to Figure 19:

$$f(x, y, \theta) = d + \cos\theta + \sqrt{(x - \cos\theta)^2 + (y - \sin\theta)^2}$$

We can use for the light rays Fermat's principle $f' = 0$, and the bifurcation set, the caustic, can be again studied by analysing the different numbers of extrema. In this case, θ is the state variable and x and y are the control parameters. The results are shown in Figure 20, clearly showing the cusp like structure.

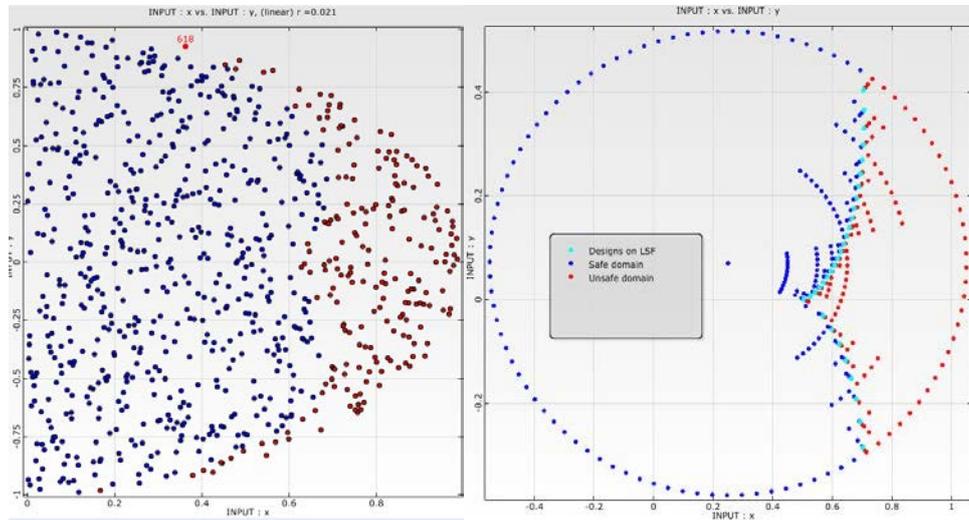


Figure 20: Anthill Plot from sensitivity analysis with color coding for the different number of extrema (left) and directional sampling with the number of extrema ≤ 1 as a limit state function (right) for detecting the cusp like caustic

Structural Mechanics

Structural Mechanics has a long history in studying stability, especially the examples of Euler, some published already in the year 1744, are examples well known in civil engineering. We take as an example the Euler truss shown in Figure 21 (length 1 for each bar).

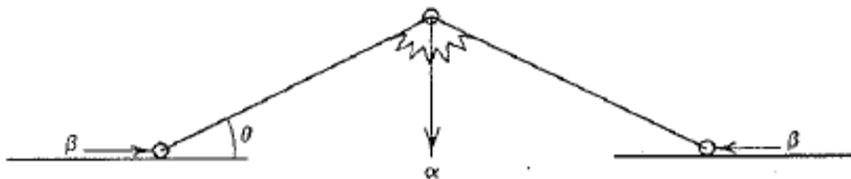


Figure 21: The Euler truss

The potential energy is the sum of the three components: spring energy with Young's modulus μ , energy due to the load force α on the top, and the energy from the displacements by the horizontal forces β :

$$V(\theta) = 2\mu\theta^2 + \alpha \sin\theta - 2\beta(1 - \cos\theta)$$

For $\alpha = 0$ the equilibrium condition $V'(\theta) = 0$ leads to

$$\beta \sin\theta = 2\mu\theta$$

For $\beta < 2\mu$ the only solution is $\theta = 0$, while for $\beta > 2\mu$ there are two additional solutions, one with positive θ and one with negative. This is related to the effect, that until β exceeds 2μ the truss remains on the ground.

For small values of $\beta - 2\mu$ and α it can be shown that the potential has the form of the cusp, essentially with the control parameters $(\beta - 2\mu)$ and α and there is a critical force, where the truss will have a sudden equilibrium change being overloaded (Saunders 1980):

$$\alpha_{crit} = \frac{10}{3\sqrt{\mu}} (\beta - 2\mu)^{\frac{3}{2}}$$

In our sensitivity analysis we used $\beta - 2\mu$ as one parameter (parameter `b_2m` in Figure 22), μ as another parameter, and β as a dependent parameter. Results of the sensitivity analysis for small and larger ranges for α are shown. Again the curves $V(\theta)$ are simply differentiated by their number of extrema. We see from the left picture that for a small range of α the critical force α_{crit} is a function increasing with $\beta - 2\mu$, however for a larger range (right picture) other characteristics appear. The approach of using the number of extrema for the limit state function is also possible for the larger range to analyse the critical force as a function of β and μ .

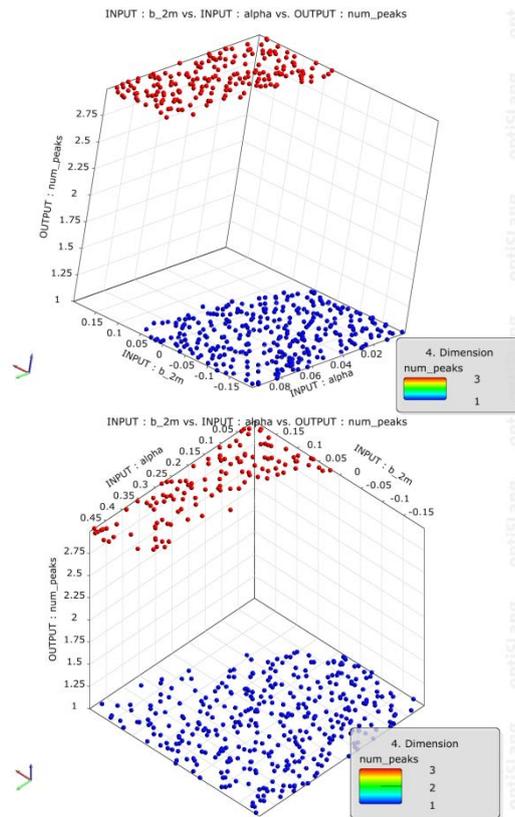


Figure 22: Sensitivity analysis with smaller (left) and larger range (right) of the force α

5. Conclusions and Outlook

In this paper we have concentrated to study some of the elementary functions of catastrophe theory with RDO techniques. The intention was to start to provide some background and a basic framework that we can use to analyse real world bifurcation scenarios. Even though many real world examples, for example a large range of caustics, can be covered by the elementary catastrophes, many other real world examples will also need the extension of this framework to higher dimensions. Also having more than one state variable will lead naturally to approaches including random field techniques.

There are many new approaches imaginable, like for example systematic studies of the design space searching for bifurcation points, to look for new solutions or studies for stabilizing the product by defining some critical area of the bifurcation set as limit state and to demand some distance of the product in design space from this limit state. Applying reliability methods from RDO to the approach of structural stability of catastrophe theory can eventually lead to structural reliability.

6. References

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